Journal of Engineering Mathematics 20 (1985) 163–169

A note on the inner problem for head-sea diffraction by a slender body of rectangular cross-section

P. McIVER

Department of Mathematics and Statistics, Brunel University, Uxbridge, Middlesex UB8 3PH, England

(Received September 13, 1985)

Summary

When the problem of the diffraction of head seas by a slender body is solved by the method of matched asymptotic expansions it is known that the inner (cross-sectional) problem can be singular at a number of discrete frequencies, though the precise circumstances under which these singularities occur is not yet clear. Here, a body of rectangular cross-section is considered, a case for which an accurate solution can be readily calculated. For non-zero draught it is found that there are two frequencies of the incident wave field for which the problem is singular, unless the draught-to-beam ratio is sufficiently large, in which case there are none.

1. Introduction

Haren and Mei [3] have considered the diffraction of head-seas by a slender body of zero draught using the method of matched asymptotic solutions. They found numerically that the cross-sectional, or inner, problem possessed a singularity at a certain frequency dependent upon the beam of the body. This singularity corresponds to a non-uniqueness of the solution to the inner problem. The solution to the full problem remains finite though Haren and Mei did report a loss of accuracy close to this frequency. Subsequently, Yue and Mei [5] showed that the singularity is inherent in the mathematical problem, rather than a property of the method of solution as are the irregular frequencies of integral-equation methods. Yue and Mei showed that for a body of non-zero draft in shallow water the inner problem is singular at a single frequency and the solution has a simple pole there. They went on to consider bodies of zero draft in arbitrary finite-depth water and found similar behaviour.

Rectangular bodies of non-zero draft have been considered to some extent by Aranha and Sugaya [1]. They found that, when they occur at all, the number of singular frequencies must be even and that there are no such frequencies if the draught-to-beam ratio is sufficiently large. Aranha and Sugaya [1] interpreted the singular frequencies by considering the full problem for a body of constant cross-section. Ursell [4] had found previously that at low frequencies the waves are diffracted away from the body leaving a comparatively wave-free zone along the body at sufficiently large distances from the bow. Aranha and Sugaya [1] have shown that for a range of frequencies immediately above the first singular frequency the solution remains oscillatory along the length of the body. Indeed, as each singular frequency is crossed there is a reversal in the type of behaviour.

The inner problem was solved by Aranha and Sugaya [1] using a hybrid-element method, which has the advantage that it may be applied to any cross-sectional shape. They presented a few results for a rectangular cross-section, however, for this geometry a more accurate solution may be obtained by the method of matched eigenfunction expansions. In this note this method is used to locate the singular frequencies as the geometry of the cross-section is varied. The location of these singularities is of interest to those proposing to carry out computations of the full diffraction problem because, as reported by Haren and Mei [3], the accuracy of the solution may be affected.

The main results of the present work are that, for a body of rectangular cross-section, there are at most two singular frequencies and there are none whenever the draught-to-beam ratio exceeds 0.1.

2. Governing equations and solution procedure

The decomposition of the full diffraction problem into inner and outer problems is fully described by Haren and Mei [3]. For waves of frequency ω and wavenumber k advancing in the direction of x increasing the total potential $\varphi(x, y, z)$ is written as

$$\varphi = \exp(ikx).\{\cosh k(z+h) + \psi(x, y, z)\} \tag{1}$$

where ψ describes the modification to the incident wave. In the scheme of matched asymptotics used by Haren and Mei [3] the inner problem is two-dimensional and to be solved at each different cross section. The equations for the inner problem are given by Yue and Mei [5]. For a rectangular body of draught D and beam $2b$ (Fig. 1) the inner problem potential must satisfy

$$\psi_{yy} + \psi_{zz} - k^2\psi = 0, \quad -h < z < 0, \; |y| > b, \tag{2a}$$

$$-h < z < -D, \; |y| < b;$$

$$\psi_z - \frac{\omega^2}{g}\psi = 0, \quad z = 0, \quad |y| > b; \tag{2b}$$

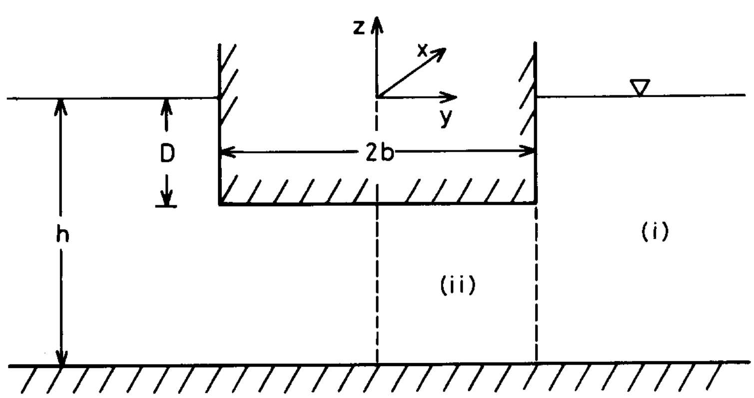

Figure 1. Definition sketch for the inner problem with a rectangular cross-section.

$$\psi_z = 0, \quad z = -h; \tag{2c}$$

$$\psi_y = 0, \quad -D < z < 0, \quad |y| = b; \tag{2d}$$

$$\psi_z = -k \sinh k(h-D), \quad z = -D, \quad |y| < b; \tag{2e}$$

and, for some constant C_0,

$$\psi \to C_0 \frac{|y|}{b} \cosh k(z+h), \quad |ky| \to \infty. \tag{21}$$

The potential in each of the two regions labelled in Fig. 1 may be represented by an eigenfunction expansion (see Yue and Mei [5]). The problem is symmetric about $y = 0$ and so the expansion will be given for $y > 0$ only. Thus, for region 1,

$$\psi_1(y, z) = A_0 ky F_{10}(z) - \sum_{n=1}^{\infty} A_n \exp(-r_n(y-b)) F_{1n}(z), \quad y > b, \tag{3a}$$

and, in region 2,

$$\psi_2(y, z) = -\cosh k(z+h) + \sum_{n=0}^{\infty} \frac{B_n \cosh q_n y\, F_{2n}(z)}{\cosh q_n b}, \quad y < b, \tag{3b}$$

where the factor $\cosh q_n b$ has been introduced for later convenience. The vertical eigenfunctions for region 1 are defined by

$$F_{1n}(z) = N_{1n}^{-1} \cos k_n(z+h), \quad n = 0, 1, 2, \ldots, \tag{4a}$$

where

$$N_{1n}^2 = \frac{1}{2}\left(h - \frac{1}{K} \sin^2 k_n h\right), \tag{4b}$$

$$r_n^2 = k_n^2 + k^2, \tag{4c}$$

and the $k_n (n \neq 0)$ are the positive real roots of

$$\frac{\omega^2}{g} = -k_n \tan k_n h, \tag{4d}$$

while $k_0 = ik$ is one of the two imaginary roots. The vertical eigenfunctions for region 2 are

$$F_{20}(z) = (h-D)^{-1/2}, \tag{5a}$$

$$F_{2n}(z) = (2/(h-D))^{1/2} \cos p_n(z+h), \quad n \neq 0, \tag{5b}$$

where

$$q_n^2 = p_n^2 + k^2, \quad n = 0, 1, 2 \ldots, \tag{5c}$$

and

$$p_n = \frac{n\pi}{h - D}, \quad n = 0, 1, 2 \ldots . \tag{5d}$$

The expansions for ψ_1 and ψ_2 satisfy all of equations (2) with the exception of (2d), this latter equation is incorporated into the matching procedure. The solution proceeds in a standard way: ψ_1 and ψ_2 are matched on $y = b$ and then the orthogonality properties of $F_{1n}(z)$ and $F_{2n}(z)$ are used to obtain two sets of equations linking the A_n and B_n. Full details of this type of procedure are given by Evans and McIver [2]. Either set of coefficients may be eliminated to obtain a single set of equations. However, the convergence properties of the systems are such that it is more advantageous to work with that for the B_n. The equations are

$$B_m + \sum_{n=0}^{\infty} U_{mn} B_n = V_m, \quad m = 0, 1, 2 \ldots, \tag{6}$$

where

$$U_{mn} = q_n b \tanh q_n b \left(-C_{0m} C_{0n} + \sum_{j=1}^{\infty} \frac{C_{jm} C_{jn}}{r_j b} \right), \tag{7a}$$

$$V_m = N_{10} C_{0m}, \tag{7b}$$

and

$$C_{mn} = \frac{1}{h} \int_{-h}^{-D} F_{1m}(z) F_{2n}(z) \, \mathrm{d}z. \tag{7c}$$

The set of equations (6) has a unique solution provided $\det(I + U)$ is not equal to zero; the singular solutions occur when the system has no unique solution, that is, if $\det(I + U)$ equals zero. If there is no incident wave train, then V_m is identically zero for all m. Therefore singular solutions of the inner problem occur at those frequencies for which the homogeneous problem, with no forcing from the incident wave, has a solution.

3. Results

The identification of the singular frequencies of the inner problem has been reduced to locating the zeros of an infinite determinant. This is done by first truncating the system (6) at a finite value of m and examining the behaviour numerically to find the zeros of the truncated determinant. The procedure is then repeated for a number of other truncations and the results extrapolated to give the location of the zeros of the infinite determinant. A discussion of the precautions required in locating the zeros of an infinite determinant of this type may be found in Evans and McIver [2].

Results for the rectangular cross-section of zero draught are given by Yue and Mei [5] and precise agreement with their values is given by the present method. In the following, the singular frequencies will be identified by their corresponding wavenumber k'. In Fig.

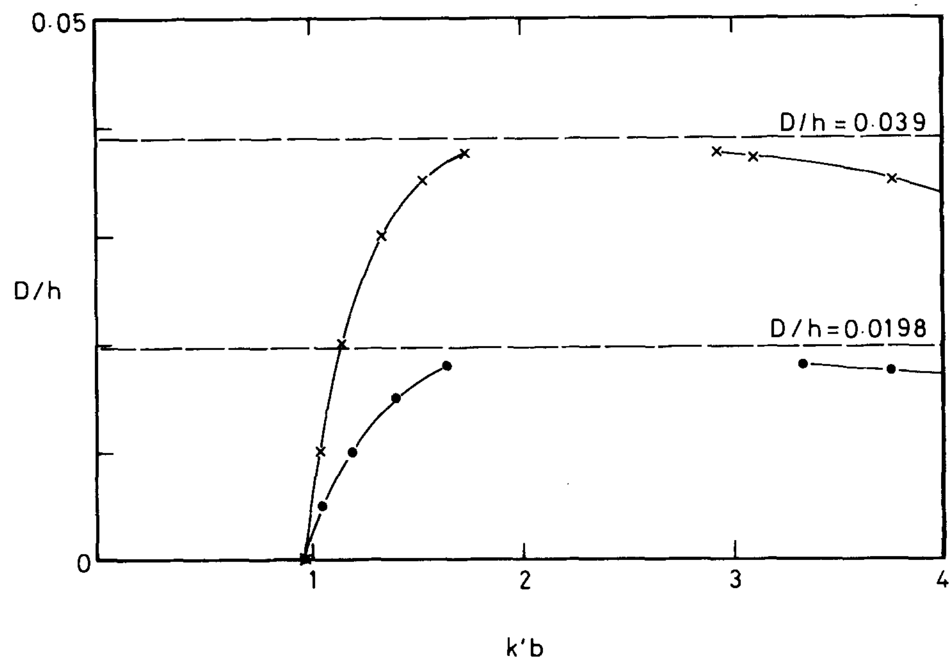

Figure 2. The relationship between the draught D/h and $k'b$ for beams $b/h = 0.1$ (●●●) and $b/h = 0.2$ (× × ×). The dashed lines indicate the maximum values of D/h for which the problem is singular with the given values of the beam.

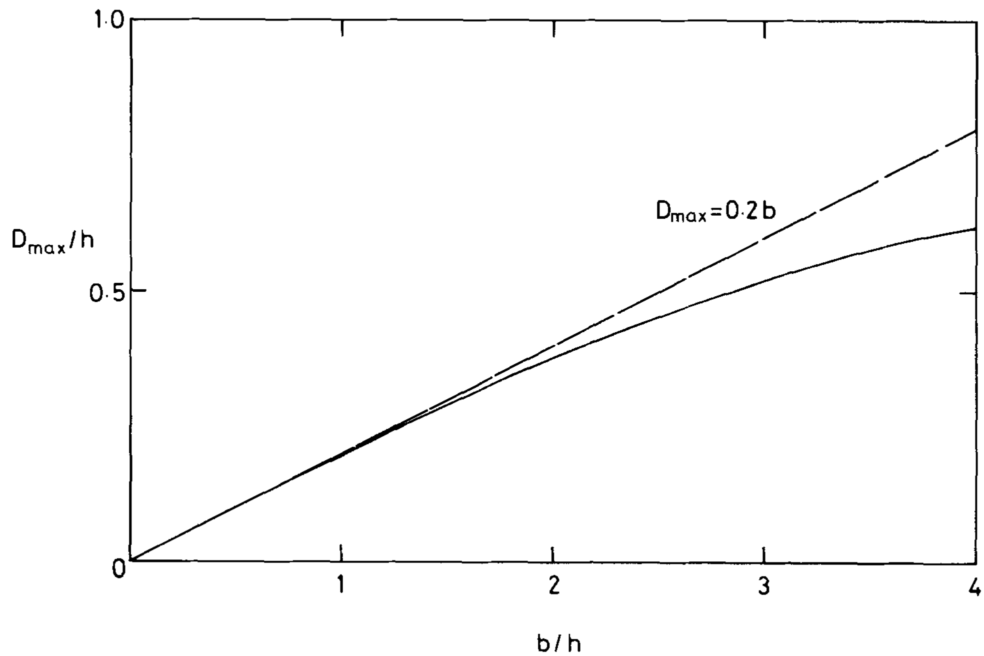

Figure 3. The maximum value D_{max}/h of the draught for which the inner problem is singular as a function of the beam b/h.

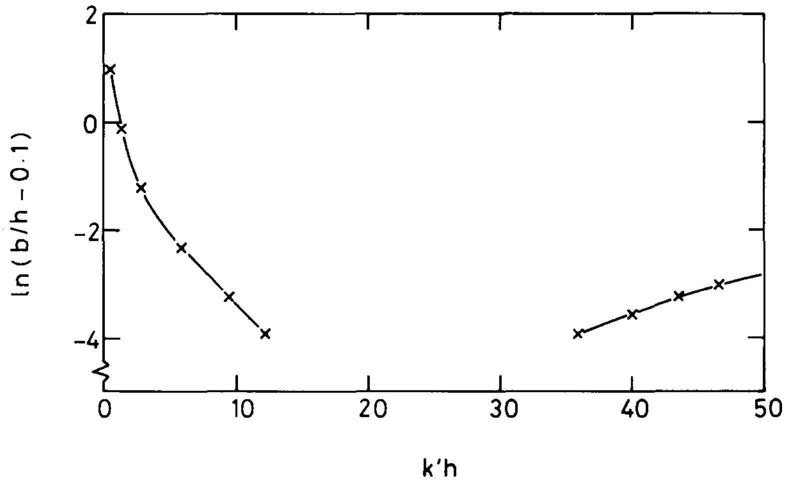

Figure 4. The relationship between the beam b/h and $k'h$ for fixed draught of $D/h = 0.02$. As b/h is very close to 0.1 over a large range of wavenumbers the natural logarithm of $(b/h - 0.1)$ has been used as ordinate.

2 the values of $k'b$, as the draught D varies, are given for two fixed values of the beam. As in all of the geometries investigated, no more than two wavenumbers were found for which the inner problem becomes singular. Below a limiting value of $D/h(=D_{max}/h$, say), there is a pair of singular frequencies for each value of D/h, at D_{max}/h the two singular frequencies appear to coalesce. Close to D_{max}/h convergence of the system is poor and the precise values of $k'b$ are difficult to determine, though the value of D_{max}/h itself is readily found by a similar extrapolation procedure as that used for the zeros of the determinant. The numerical evidence suggests that for small b/h then D_{max}/b is approximately 0.2. From the shallow-water theory of Yue and Mei [5] it can be seen that D_{max}/h must approach unity as b/h tends to infinity. The intermediate behaviour is shown in Fig. 3. It is apparent that there are no singular frequencies whenever the draught-to-beam ratio $D/2b$ is greater than 0.1, whatever the beam of the body. Aranha and Sugaya [1] give an upper bound on $D/2b$ of approximately 0.26 for the existence of singular frequencies.

As D/h tends to zero the limiting wavenumbers $k'b$, shown in Fig. 2 for two values of b/h, are distinct. For $b/h = 0.1$ then $k'b$ tends to 0.993, while for $b/h = 0.2$, $k'b$ tends to 0.994. The two curves cross close to $D/h = 0$. As b/h increases, the value of $k'b$ at $D/h = 0$ increases to the shallow-water limit (see Yue and Mel [5]) of approximately 2.4. Figure 4 shows the values of $k'h$ for fixed draught as the beam varies. In this case with a fixed draught of $D = 0.02$ there are no singular frequencies for b/h less than about 0.1. As the beam becomes a significant proportion of the depth the singular behaviour occurs only for very long or very short waves.

Acknowledgement

The author is grateful to Dr D.V. Evans for suggesting this work and the Science and Engineering Research Council for financial support.

References

[1] Aranha, J.A.P. and Sugaya, M., A note on head-sea diffraction by a slender body, *J. Fluid Mech.* 155 (1985) 131–139.

[2] Evans, D.V. and McIver, P., Edge waves over a shelf: full linear theory, *J. Fluid Mech.* 142 (1984) 79–95.

[3] Haren, P. and Mei, C.C., Head-sea diffraction by a slender raft with application to wave-power absorption, *J. Fluid Mech.* 104 (1981) 505–526.

[4] Ursell, F., The refraction of head-seas by a long ship; Part 2: Waves of long wavelength, *J. Fluid Mech.* 82 (1977) 643–657.

[5] Yue, D.K.P. and Mei, C.C., A note on the singularity of an inner problem for head-sea diffraction by a slender body, *J. Fluid Mech.* 109 (1981) 253–256.